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Abstract. Here we consider discrete dynamical systems on the unit interval. We discuss the quadratic family and a family of sine maps, and analyse the dynamical systems generated by these maps, both numerically and analytically.

I. Introduction

We use the following notation throughout the report: $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ is the set of integers, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+ = \{\, x \in \mathbb{R} \mid x > 0 \,\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

I.I. **Preliminaries.**

Definition 1. A dynamical system is a triple (X, T, Φ) where X is a nonempty set, $T \subseteq \mathbb{R}$ is an additive monoid and $\Phi : X \times T \to X$ is a function that satisfies $\Phi : (x, 0) \mapsto x$ and $\Phi : \big(\Phi(x,t), \tau\big) \mapsto \Phi(x, t+\tau)$ for all $x \in X$ and for all $t, \tau \in T$. We call a dynamical system with $T \subseteq \mathbb{Z}$ a *discrete* dynamical system.

Remark 2. Consider a dynamical system (X, T, Φ) with $T = \mathbb{R}$ (resp. $T = \mathbb{R}_0^+$). Such a system is clearly non-discrete. Now let $\mathcal{T} = \mathbb{Z}$ (resp. $\mathcal{T} = \mathbb{N}_0$) and define $\varphi : X \times \mathcal{T} \to X$ by $(x, n) \mapsto \Phi(x, hn)$, where $h \in \mathbb{R}^+$. Then $(X, \mathcal{T}, \varphi)$ is a discrete dynamical system.

Definition 3. Consider a non-discrete dynamical system (V, T, Φ) where V is a vector space and Φ is of class C^2. The vector field associated with the dynamical system is $C^1 \ni f : V \to V$ by $x \mapsto D_t\Phi(x, 0)$.

Proposition 1. *Let f be the vector field associated with the dynamical system (V, T, Φ) of definition 3. Then Φ is a solution of the differential equation*

$$Dx(t) = f\big(x(t)\big). \tag{1}$$

Proof. Let $x(t) = \Phi(s, t)$. Now, using the group properties of Φ,

$$\begin{aligned} Dx(t) &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big[\Phi(s, t+\epsilon) - \Phi(s,t)\big] \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big[\Phi\big(x(t), \epsilon\big) - \Phi\big(x(t), 0\big)\big] \\ &= f\big(x(t)\big), \end{aligned}$$

and Φ is a solution of eq. (1). ∎

Remark 4 (Euler method). Let (V, T, Φ) be the dynamical system of definition 3 and suppose that the function Φ is determined by the differential equation $Dx(t) = f\big(x(t)\big)$. By remark 2 we may approximate Φ with φ. Now approximate φ by retaining only the first two terms of its Maclaurin series, i.e., let $\varphi(x, 1) = \Phi(x, h) = \Phi(x, 0) + hD_t\Phi(x, 0) = x + hf(x)$. Then φ and its iterates provide a simple way to approximate solutions of eq. (1), with increasing accuracy for decreasing step size h.

Now consider the dynamical system (X, T, Φ) of definition 1.

Definition 5. We say a subset $A \subseteq X$ is Φ-invariant if $\Phi(A, T) \subseteq A$.

Definition 6. We say a point $x \in X$ is *periodic* if $\Phi(x, t) = x$ for some $t \in T_{>0}$, and, if such a t exists, we call it a *period* of x. Obviously, for any period t of x, all elements of $t\mathbb{N}$ are periods of x. The smallest period of x (should it exist) is called the *prime period* of x.

Definition 7. The orbit of $x \in X$ under Φ is the sequence $\mathcal{O}_\Phi(x) := \big(\Phi(x,t)\big)_{t \in T}$. If x is periodic with period p, we say the orbit of x is p-periodic, or a p-cycle.

Definition 8. Let $x \in X$. If $\Phi(x, T) = \{x\}$, the point x is a *fixed point* of the dynamical system (X, T, Φ). This is equivalent to saying the orbit of x under Φ is 1-periodic, or a 1-cycle.

Example 9. Consider the dynamical system $(X, \mathbb{R}, \Phi)$ and suppose that $x \in X$ is a fixed point of the system. Then every $t \in \mathbb{R}^+$ is a period of x but there exists no prime period of x.

Definition 10. Let (X, T, Φ) be a dynamical system where X is a Hausdorff space and Φ is continuous. Suppose that $x \in X$ is a fixed point. We say the point x is *stable* if, for all neighbourhoods $\mathcal{U}$ of x, there exists a neighbourhood $\mathcal{V}$ of x such that $\Phi(\mathcal{V}, t) \subseteq \mathcal{U}$ for all $t \in T_{\geq 0}$. If the point x is not stable, we say it is *unstable.*

Remark 11. To distinguish between other possible definitions of stability, a stable fixed point, as defined in definition 10, is often referred to as Lyapunov stable, or Poisson stable. Other notions of stability may require, e.g., that there exists a neighbourhood of the fixed point such that the orbits of its points converge to the fixed point. In this case the stable fixed point may be referred to as asymptotically stable, or an attracting fixed point.

Let (X, T, Φ) be a discrete dynamical system where $X \subseteq \mathbb{R}$ and the function Φ is of class C^1. Note that $\Phi(x, n) = \Phi^n(x)$ for all $x \in X$, where Φ^n is the n-fold composition of Φ and $\Phi^0 := \mathrm{id}_X$. Since for $r, s \in X$

$$\Phi^n(r) = \Phi^n(s) + (r-s)D\Phi^n(s) + O\big[(r-s)^2\big],$$

we have, after n iterations and to first order in $r - s$,

$$\begin{aligned}\Phi^n(r) - \Phi^n(s) &= (r-s)D\Phi^n(s) \\ &= (r-s)\prod_{\ell=0}^{n-1} D\Phi^1\big(\Phi^\ell(s)\big).\end{aligned}$$

Letting $r \to s$ the asymptotic equality

$$|\Phi^n(r) - \Phi^n(s)| \sim e^{n\chi(s)}|r-s|$$

holds as $n \to \infty$. Here χ is called the *Lyapunov exponent,* defined as follows.

Definition 12. Let (X, T, Φ) be a discrete dynamical system with $X \subseteq \mathbb{R}$ and $\Phi \in C^1$. The Lyapunov exponent for $s \in X$ is

$$\begin{aligned}\chi(s) &:= \lim_{n\to\infty} \frac{1}{n} \log \left| \prod_{\ell=0}^{n-1} D\Phi^1\big(\Phi^\ell(s)\big) \right| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{\ell=0}^{n-1} \log \big| D\Phi^1\big(\Phi^\ell(s)\big)\big|,\end{aligned}$$

assuming the limit exists. Here Φ^n is the n-fold composition of Φ and $\Phi^0 := \mathrm{id}_X$.

The Lyapunov exponent χ represents the average exponential rate of divergence of two orbits initially infinitesimally close to each other, as $n \to \infty$.

Remark 13. While we have defined the Lyapunov exponent for a discrete one-dimensional dynamical system, the notion exists for non-discrete as well as multidimensional dynamical systems. See any advanced textbook on dynamical systems or differential equations for a definition. A discussion of literature that helped shape this report is presented in appendix B.

Definition 14. A dynamical system $(Y \subseteq \mathbb{R}, T, \Phi)$ has sensitive dependence on initial conditions (SDIC) on a subset $X \subseteq Y$ if there exists $\delta > 0$ such that for every $x \in X$ and $\epsilon > 0$ there exists $y \in Y$ and $t \in T_{>0}$ for which $|x - y| < \epsilon$ and $|\Phi(x, t) - \Phi(y, t)| > \delta$.

While there is no universally agreed definition of chaos, *chaotic dynamical systems* are generally defined as systems with SDIC, usually in addition to other properties (such as the existence of a dense orbit). Most dynamical systems with positive Lyapunov exponents exhibit SDIC. Moreover, most dynamical systems with SDIC are expected to have positive Lyapunov exponents. (Brin & Stuck 2004; see Balibrea & Victoria Caballero 2014 for an example of a dynamical system that has a positive Lyapunov exponent but no SDIC. They also construct a dynamical system that has a negative Lyapunov exponent and SDIC.)

I.II. **The Quadratic Family.**

Definition 15. The logistic map[1] is the map $f_\lambda : \mathbb{R} \to \mathbb{R}$ by $x \mapsto \lambda x(1 - x)$, where $\lambda \in \mathbb{R}$ is a parameter.

Proposition 2. *Let $F_\lambda : \mathbb{R} \times \mathbb{N}_0 \to \mathbb{R}$ by $(x, n) \mapsto f_\lambda^n(x)$. The unit interval $\mathbb{I} := [\,0, 1\,]$ is F_λ-invariant for all $\lambda \in [\,0, 4\,]$.*

Proof. The logistic map f_λ is unimodal[2] with zeroes at $x \in \{0, 1\}$ and a maximum value $\lambda/4$ at $x = 1/2$. It follows that $F_\lambda(\mathbb{I}, \mathbb{N}_0) \subseteq \mathbb{I}$ for all $\lambda \in [\,0, 4\,]$. ∎

Definition 16. The family of logistic maps $\{\, f_\lambda : \mathbb{I} \to \mathbb{I} \mid \lambda \in [\,0, 4\,] \,\}$ is called the *quadratic family.*

I.III. **The Sine Maps.**

Definition 17. The sine map is the map $f_\mu : \mathbb{R} \to \mathbb{R}$ by $x \mapsto \mu \sin(\pi x)$, where $\mu \in \mathbb{R}$ is a parameter.

Proposition 3. *Let $F_\mu : \mathbb{R} \times \mathbb{N}_0 \to \mathbb{R}$ by $(x, n) \mapsto f_\mu^n(x)$. The unit interval $\mathbb{I}$ is F_μ-invariant for all $\mu \in [\,0, 1\,]$.*

Proof. The sine map f_μ is unimodal with zeroes at $x \in \{0, 1\}$. The maximum value is μ at $x = 1/2$ and, clearly, $F_\mu(\mathbb{I}, \mathbb{N}_0) \subseteq \mathbb{I}$ for all $\mu \in [\,0, 1\,]$. ∎

II. Methods

II.I. **Stability Analysis of the Quadratic Family.**

Consider the quadratic family of definition 16 and let $F_\lambda : \mathbb{I} \times \mathbb{N}_0 \to \mathbb{I}$ by $(x, n) \mapsto f_\lambda^n(x)$. For $x \in \mathbb{I}$ to be a fixed point of the dynamical system $\mathcal{Q} := (\mathbb{I}, \mathbb{N}_0, F_\lambda)$ we must have $f_\lambda(x) = \lambda x(1 - x) = x$. Therefore, the fixed points of the dynamical system are 0 and $1 - 1/\lambda$. Note that, since $1 - 1/\lambda \notin \mathbb{I}$ for $\lambda \in [\,0, 1)$, the nonzero fixed point exists only for $\lambda \in [1, 4\,]$.

Proposition 4. *The fixed point 0 of the dynamical system $\mathcal{Q}$ is stable for all $\lambda \in [\,0, 1\,]$ and unstable for all $\lambda \in (1, 4\,]$.*

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[1] We refer to functions with equal domain and codomain as maps.

[2] We say a function $f : X \to Y$ is unimodal if there exists $c \in X$ such that f is monotonically increasing for $X \ni x \leq c$ and monotonically decreasing for $x \geq c$.

Proof. First let $\lambda \in [0,1]$ and $\delta \in (0,1]$. The interval $[0,\delta]$ is a neighbourhood of 0 and we have $f_\lambda(x) = \lambda x(1-x) \le x(1-x) \le x$ for all $x \in [0,\delta]$. By induction the interval $[0,\delta]$ is F_λ-invariant and thus the fixed point 0 is stable for all $\lambda \in [0,1]$.

Then let $\lambda \in (1,4]$ and consider the interval $[0, 1-1/\lambda)$. For every $x \in (0, 1-1/\lambda)$ we have $f_\lambda(x) = \lambda x(1-x) > x$. It follows immediately that the fixed point 0 is unstable for all $\lambda \in (1,4]$. ∎

Lemma 1. *Let (X, T, Φ) be a discrete dynamical system with $X \subseteq \mathbb{R}$ and the function Φ of class C^1. Suppose that $p \in X$ is a fixed point of the dynamical system and let $\phi : x \mapsto \Phi(x,1)$ for all $x \in X$. Then the fixed point p is stable if $|D\phi(p)| < 1$, and unstable if $|D\phi(p)| > 1$.*

Proof. First suppose that $|D\phi(p)| =: \vartheta < 1$ and let $k \in (\vartheta, 1)$. By continuity of $D\phi$ there exists $\delta > 0$ such that $|D\phi(x)| < k$ for all $x \in [p-\delta, p+\delta] =: I$. Now let $x \in I$. By the mean value theorem there exists c between x and p such that $\phi(x) - \phi(p) = (x-p)D\phi(c)$. Thus $|\phi(x) - p| < |x-p|$ and the interval I is Φ-invariant showing that the fixed point p is stable.

The proof of the second statement is completely analogous. ∎

Corollary 1. *It is readily verified that if $|D\phi(p)| < 1$ the orbit of every point on the interval I converges exponentially to the fixed point p.*

Remark 18. In lemma 1 we did not discuss the case $|D\phi(p)| = 1$. Indeed, in this case we can not make any conclusions on the stability of the fixed point p and must resort to other methods.

Lemma 2. *Consider a discrete dynamical system (I, T, Φ) where $I := [\alpha, \beta] \subset \mathbb{R}$ is a closed bounded interval and Φ is a continuous nondecreasing map that has no fixed points in (α, β). Then α or β is a fixed point of Φ and the orbit of every point on the interval I converges to it, except the orbit of the other endpoint if it is a fixed point as well.*

Proof. Let $\phi : x \mapsto \Phi(x,1)$ for all $x \in X$. Now $\phi(\alpha) \ge \alpha \Leftrightarrow (\phi - \mathrm{id})(\alpha) \ge 0$ and $\phi(\beta) \le \beta \Leftrightarrow (\phi - \mathrm{id})(\beta) \le 0$. By assumption $\phi - \mathrm{id}$ is never zero on (α, β), and by the intermediate value theorem it cannot change sign on I. Hence either $\phi(\alpha) = \alpha$ or $\phi(\beta) = \beta$. Assume without loss of generality that $\phi(\alpha) = \alpha$, i.e., α is a fixed point. Then $\phi(x) < x$ for all $x \in (\alpha, \beta)$ and the orbit of x is decreasing and bounded below. It follows that the orbit of x is convergent. The limit is the infimum of the orbit, α, since there are no other fixed points on (α, β) by assumption. ∎

Proposition 5. *The fixed point $1 - 1/\lambda =: \xi$ of the dynamical system $\mathcal{Q}$ is stable for all $\lambda \in (1,3]$ and unstable for all $\lambda \in (3,4]$.*

Proof. Now $Df_\lambda(\xi) = 2 - \lambda$ and, by lemma 1, the fixed point ξ is stable for all $\lambda \in (1,3)$. Moreover, ξ is unstable for all $\lambda \in (3,4]$.

Let then $\lambda = 3$ and first note that the interval $[1-\xi, f_3(1/2)] = [1/3, 3/4] =: I$ is invariant under f_3. Now consider the map f_3^2. It has, in addition to the fixed point 0, the fixed point $\xi = 2/3$ (see the discussion below) and it is nondecreasing on the interval $[1/2, \xi] = [1/2, 2/3] =: J$. Note that $f_3^2(J) \subseteq J$. By lemma 2 the orbit of every point on the interval J is convergent to ξ under f_3^2. Since $f_3(I \setminus J) \subseteq J$ we have, for all $x \in I$, either $f_3^{2n}(x) \to \xi$ or $f_3^{2n+1}(x) \to \xi$ as $n \to \infty$. But this implies that $f_3^n(x) \to \xi$ as $n \to \infty$. It follows that the fixed point ξ is stable for $\lambda = 3$. ∎

Remark 19. The orbit of $x \in \mathbb{I} \setminus \{0\}$ under f_3 converges to $\xi = 2/3$ subexponentially.

We have seen that there exists no stable fixed point of the dynamical system $\mathcal{Q}$ for $\lambda \in (3,4]$. Now let us examine the behaviour of the system in this parameter range. To that end consider the dynamical system $\mathcal{Q}_1 := (\mathbb{I}, 2\mathbb{N}_0, F_\lambda)$ which is a *subsystem* of the dynamical system $\mathcal{Q}$. The fixed points of the dynamical system $\mathcal{Q}_1$ are found by solving the equation $f_\lambda^2(x) = \lambda^2 x(1-x)[1-\lambda x(1-x)] = x$. After discarding the known solutions by calculating the quotient $\big(f_\lambda^2(x) - x\big)/\big(f_\lambda(x) - x\big)$ we are left with the quadratic equation $(\lambda x)^2 - \lambda(\lambda+1)x + \lambda + 1 = 0$ which has the roots

$$\frac{1}{2\lambda}\Big(\lambda + 1 \pm \sqrt{(\lambda+1)(\lambda-3)}\Big) =: \varpi_\pm.$$

Note that for positive λ a real solution exists only for $\lambda \geq 3$ and for $\lambda = 3$ we have the single fixed point 2/3 already discussed in proposition 5. Now for $\lambda > 3$ the dynamical system $\mathcal{Q}_1$ contains two fixed points—this means the dynamical system $\mathcal{Q}$ contains a 2-cycle. So, at $\lambda = 3$ the dynamical system $\mathcal{Q}$ experiences a *bifurcation*, i.e., a change in its orbit structure since the stable 1-periodic orbits are replaced by 2-periodic orbits. Let us now determine the stability of these 2-cycles.

Proposition 6. *The fixed point $\varpi_\pm$ of the dynamical system $\mathcal{Q}_1 = (\mathbb{I}, 2\mathbb{N}_0, F_\lambda)$ is stable for all $\lambda \in (3, 1+\sqrt{6}\,]$ and unstable for all $\lambda \in (1+\sqrt{6}, 4\,]$.*

Proof. First note that $Df_\lambda^2(x) = Df_\lambda\big(f_\lambda(x)\big) \cdot Df_\lambda(x)$ and thus $Df_\lambda^2(\varpi_\pm) = Df_\lambda(\varpi_+) \cdot Df_\lambda(\varpi_-)$. Now $Df_\lambda(\varpi_\pm) = -1 \mp \sqrt{(\lambda+1)(\lambda-3)}$. Hence $Df_\lambda^2(\varpi_\pm) = 1 - (\lambda+1)(\lambda-3)$ and by lemma 1 the fixed point $\varpi_\pm$ is stable for $\lambda \in (3, 1+\sqrt{6}\,)$ and unstable for $\lambda \in (1+\sqrt{6}, 4\,]$.

We must again turn our attention to the *bifurcation point*. The construction of the proof of the stability of the fixed point for $\lambda = 1+\sqrt{6}$ is identical to the construction used in proposition 5. We will not repeat it here. ∎

It is shown in de Melo & van Strien (1993) that there exists a monotonic sequence of parameter values λ at which stable orbits with periods equal to powers of 2 emerge. This means that there exist λ_n and λ_{n+1} (here $n \in \mathbb{N}_0$) such that $\mathcal{Q}_n := (\mathbb{I}, 2^n\mathbb{N}_0, F_\lambda)$ has a stable 2^n-periodic orbit for $\lambda \in (\lambda_n, \lambda_{n+1}]$. Furthermore, this sequence converges to the point $\lambda_\infty \approx 3.57$, often called the Feigenbaum point. Moreover, the dynamical system $\mathcal{Q}_\infty$ has unstable periodic orbits of every period $p \in \mathbb{N}$, and no other periodic orbits.

The phenomenon that the distance between successive bifurcation points decreases in this manner was discovered by Feigenbaum (1978) using numerical methods. It was proved by Collet et al. (1980) that the parameter that controls the convergence is $\delta := \lim_{n\to\infty}(\lambda_n - \lambda_{n-1})/(\lambda_{n+1} - \lambda_n) \approx 4.67$, now known as the Feigenbaum constant.

Incredibly enough, the behaviour discussed above, in the case of the quadratic family, is seen in all nearby[3] unimodal maps of one parameter, and with the same *universal* parameter δ that controls the period-doubling cascade.

For $\lambda > \lambda_\infty$ we have, depending on λ, either periodic, aperiodic nonchaotic, or chaotic orbits.

II.II. **Numerical Analysis.** We wrote a computer program to evaluate the quadratic family and the sine maps. The program allows one to calculate orbits and Lyapunov exponents for given functions. For more details see appendix A.

We made 1000 iterations of the maps, using a seed value of 0.3.

II.II.I. *The Quadratic Family.* For the logistic map we used 1501 parameter values from the interval $[\,2.5, 4.0\,]$, with a spacing of 10^{-3}. In addition, we used the single parameter values 0.5 and 2.0.

[3]See section 11.3 of Hasselblatt & Katok (2003) for a more detailed discussion.

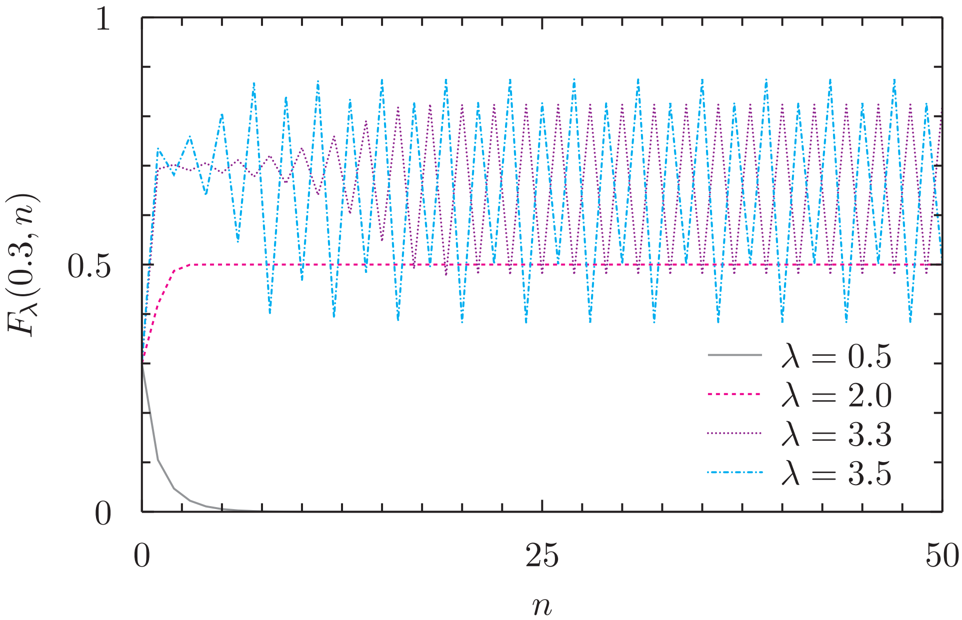

FIGURE 1. The first 51 values of $\mathcal{O}_{F_\lambda}(0.3)$ with $\lambda \in \{0.5, 2.0, 3.3, 3.5\}$.

II.II.II. *The Sine Maps.* For the sine map we used 1501 parameter values from the interval $[\,0.25, 1.0\,]$, with a spacing of $5 \cdot 10^{-4}$.

III. Results

III.I. **The Quadratic Family.** The analysis of the quadratic family in section II.I shows that the logistic map experiences a bifurcation from 1-periodic orbits to 2-periodic orbits. We discussed the existence of orbits of period $p \in \mathbb{N}$, referring to literature for proofs. This behaviour is also seen in the numerical analysis carried out in section II.II. See fig. 1 and fig. 4. The bifurcation diagram, fig. 2, has been made using the values of the last 301 iterations in order to increase the accuracy by allowing the system to relax.

The numerical analysis also confirms the correlation between positive Lyapunov exponents and chaotic behaviour, as expected. See fig. 3.

III.II. **The Sine Maps.** As already mentioned in section II.I, a period-doubling cascade occurs for all unimodal maps of one parameter that are nearby the logistic map. The numerical analysis carried out here confirms this for the family of sine maps $\{\, f_\mu : \mathbb{I} \to \mathbb{I} \mid \mu \in [\,0, 1\,]\,\}$; the dynamics are similar to those of the quadratic family. See fig. 5, fig. 8 and fig. 6. What is more, there is striking resemblance between the values of the Lyapunov exponents, see fig. 7 and cf. fig. 3.

As with the quadratic family, the bifurcation diagram uses the values of the last 301 iterations, thus providing greater accuracy by allowing the system to relax.

IV. Discussion

While we did not analyse the family of sine maps analytically, our numerical study suggests a deep connection between the sine maps and the quadratic family.

A particular notion of a connection between dynamical systems is topological conjugacy.

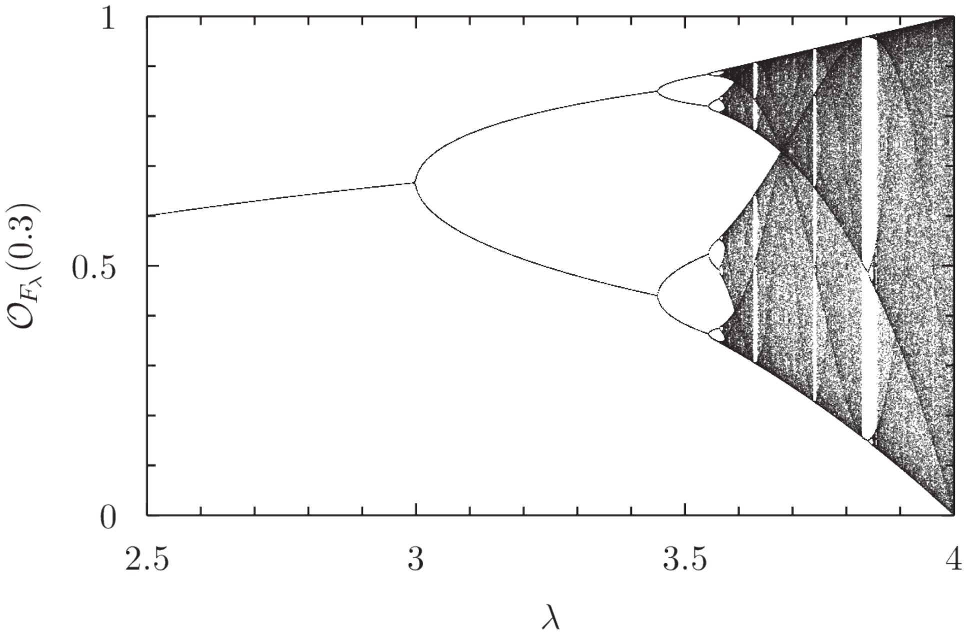

FIGURE 2. The bifurcation diagram for the logistic map with $\lambda \in [2.5, 4.0]$ and a seed of 0.3.

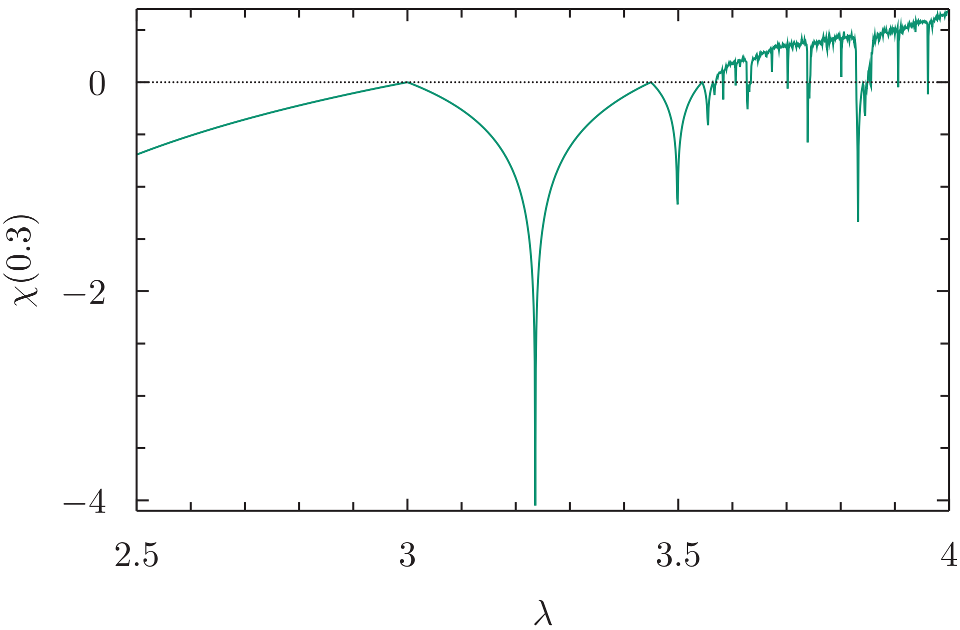

FIGURE 3. The Lyapunov exponents for the logistic map with $\lambda \in [2.5, 4.0]$ and a seed of 0.3.

Definition 20. Let (X, T, Φ) and (Y, T, Ψ) be dynamical systems where X and Y are topological spaces, and the functions Φ and Ψ are continuous. We say the dynamical systems (X, T, Φ) and (Y, T, Ψ) are topologically conjugated, or isomorphic, if there exists a homeomorphism $h : X \to Y$ such that $h\big(\Phi(x, t)\big) = \Psi\big(h(x), t\big)$ for all $x \in X$ and $t \in T$.

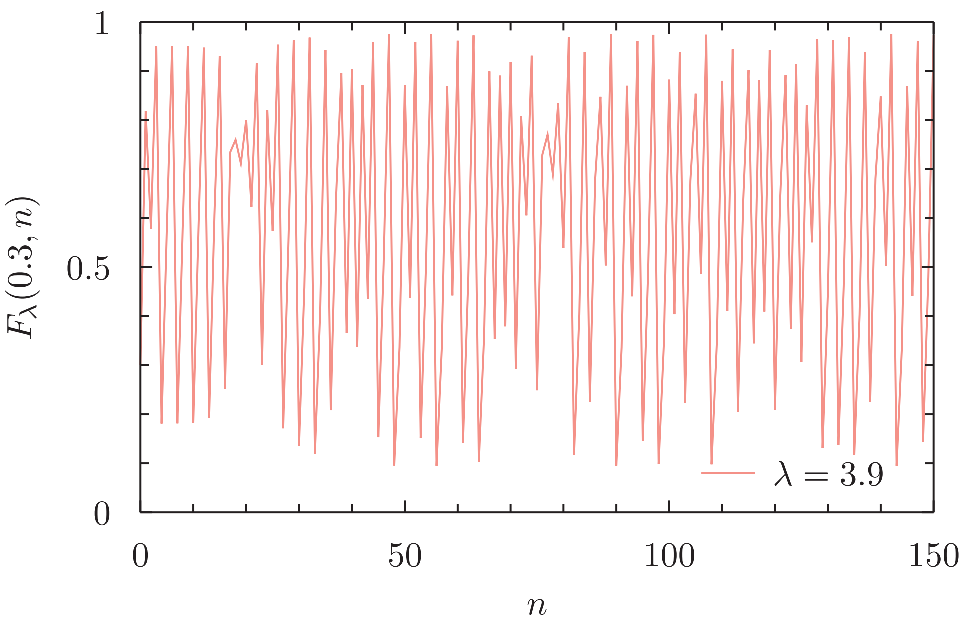

FIGURE 4. The first 151 values of $\mathcal{O}_{F_\lambda}(0.3)$ with $\lambda = 3.9$.

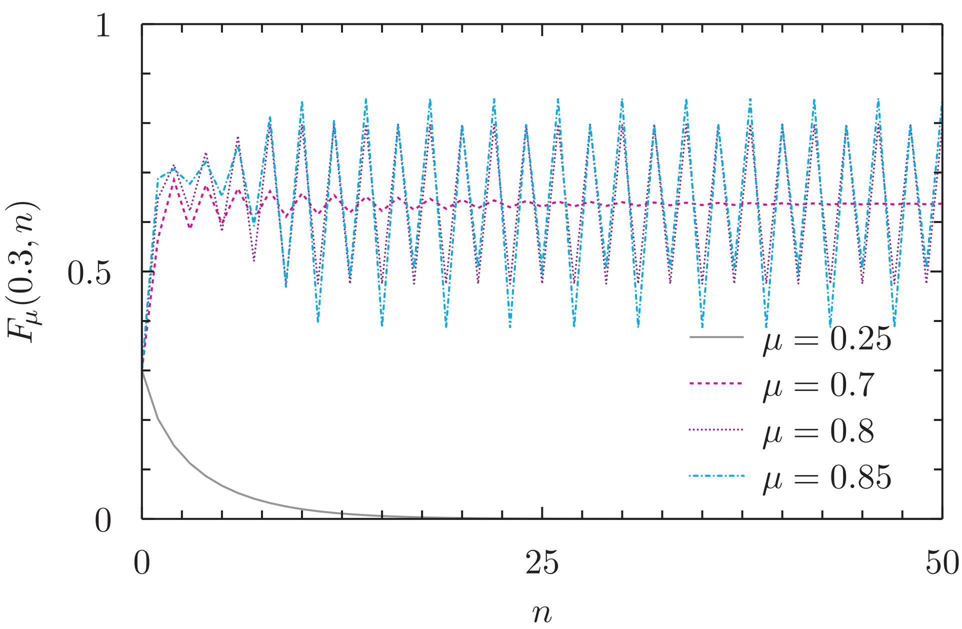

FIGURE 5. The first 51 values of $\mathcal{O}_{F_\mu}(0.3)$ with $\mu \in \{0.25, 0.7, 0.8, 0.85\}$.

One might be tempted to explore the notion of topological conjugacy with the dynamical systems considered in this report. However, finding topological conjugacies is not simple and includes many subtleties. This is beyond the scope of this report.

APPENDIX A. CODE

The code was written in Julia, version 0.5.0.

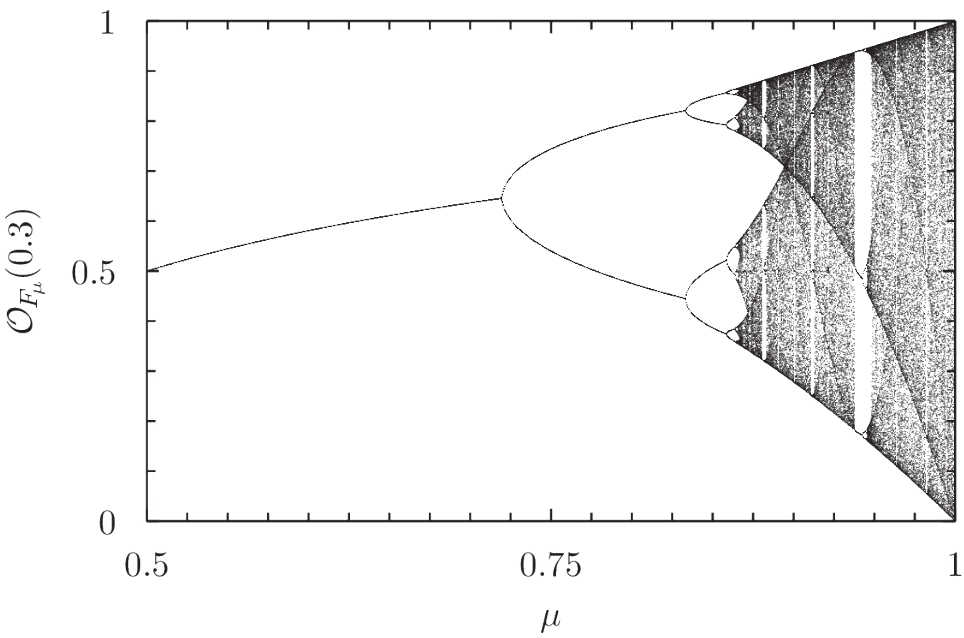

FIGURE 6. The bifurcation diagram for the sine map with $\mu \in [\,0.5, 1.0\,]$ and a seed of 0.3.

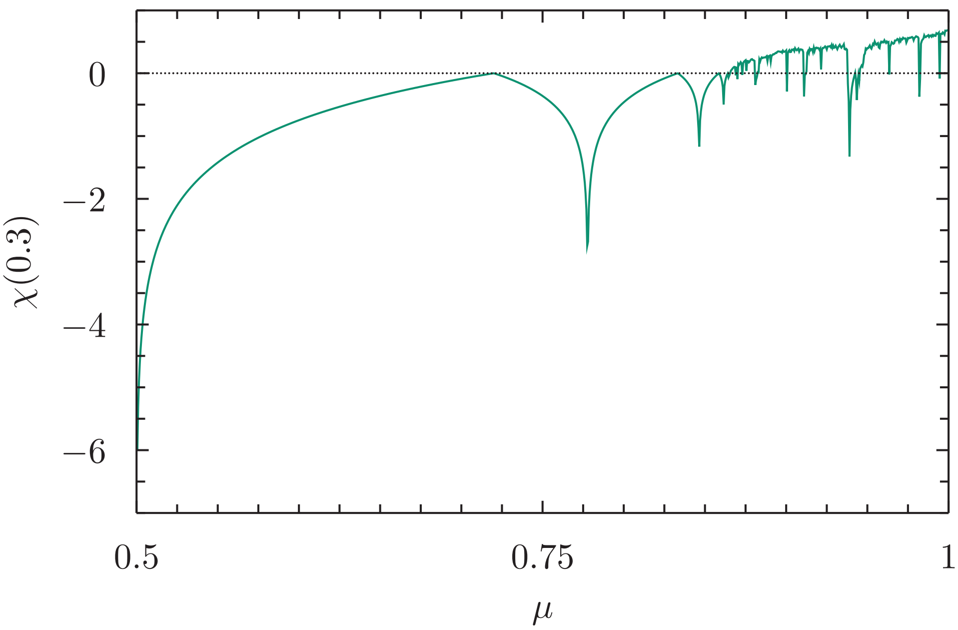

FIGURE 7. The Lyapunov exponents for the sine map with $\mu \in [\,0.5, 1.0\,]$ and a seed of 0.3.

We have defined two immutable composite types, `Q` and `R`. The type `Q` includes the function `f` (of one parameter) to be iterated, the seed value `s` for the function, and the number of iterations `n`. The type `R` includes the function `g` (logarithm of the absolute value of the derivative of `f`) needed in calculating the Lyapunov exponents (see definition 12) and an array `a` of parameters for the function `f`.

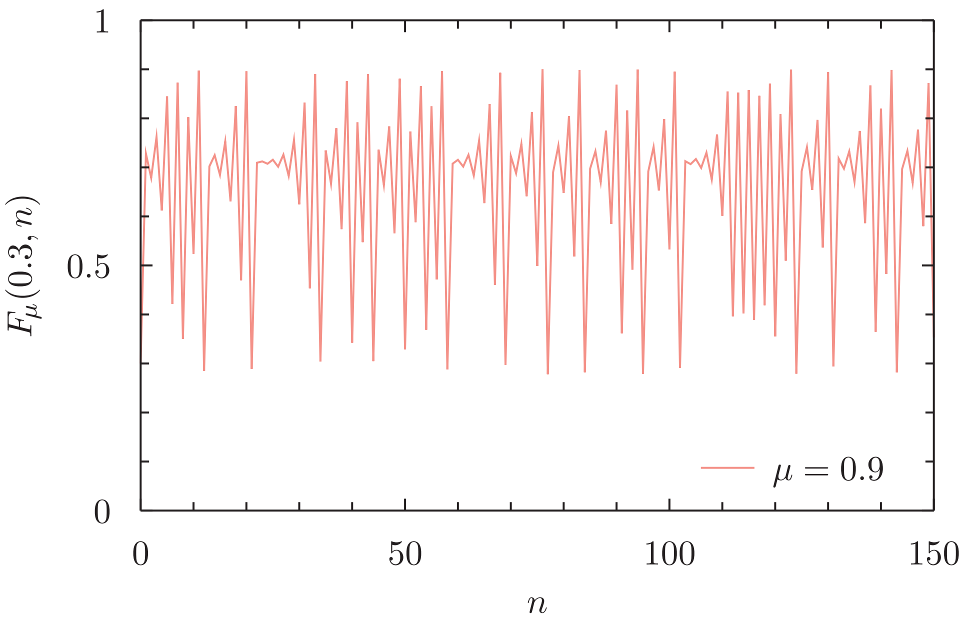

FIGURE 8. The first 151 values of $\mathcal{O}_{F_\mu}(0.3)$ with $\mu = 0.9$.

The function `orbit` calculates `n` iterates of `f` for the given seed value `s` and parameter `p`. It returns an array containing the seed value and the `n` iterates.

The function `tab` calculates the iterates of `f` for each parameter `p` of the array `a` using the function `orbit`. It also calculates the Lyapunov exponents. The results are written to a file named after the function `f`; the first column indicates the order of the iterate, the second column indicates the used parameter and the third column indicates the value of the iterate. Each parameter gets its own block, separated from other blocks by a newline. The Lyapunov exponents are written to the end of the file, two newlines after the previous block of iterate values. The block containing the Lyapunov exponents has two columns; the first column indicates the parameter value and the second indicates the corresponding value of the Lyapunov exponent (or, to be exact, an approximation of it—note that increasing the number of iterations increases the accuracy of the values). Furthermore, to increase the accuracy by allowing the system to relax, we calculate the Lyapunov exponents using only the last `n/2` iterates.

```
immutable Q
  f::Function # function
  s::Number   # seed
  n::Int      # number of iterations
end

immutable R
  g::Function # log|f'(.)|
  a::Array    # parameters
end

function orbit(q::Q,p::Number)
  orb=[q.s,zeros(q.n)...]
  for i=1:q.n
    orb[i+1]=q.f(orb[i],p)
```

```
  end; return orb
end

function tab(q::Q,r::R)
  lya=zeros(r.a)
  k=floor(Int,q.n/2)
  open("$(q.f).txt","w") do io
    for (i,p) in enumerate(r.a)
      orb=orbit(q,p)
      lya[i]=sum([r.g(o,p) for o in orb[end-k+1:end]])/k
      writedlm(io,[[0:q.n...] fill(p,1+q.n) orb])
      write(io,"\n")
    end
    write(io,"\n")
    writedlm(io,[[r.a...] lya])
  end
end
```

Appendix B. Literature & Thoughts

While we spent much time with the brilliant textbook by Hasselblatt & Katok (2003), not much of their content has influenced this report (with the exception of the proof of proposition 5). Halfway through writing the report we came across Broer & Takens (2011) and found their approach to presenting dynamical systems so logical and intuitive that we abandoned our initial approach based on Hasselblatt & Katok (2003). Consequently, the approach and notation taken here, in defining dynamical systems and their properties, was inspired by Broer & Takens (2011). We found their level of rigour enlightening. For example, the deterministic nature of dynamical systems is immediately clear from definition 1. Moreover, the notion of a dynamical system is presented in a way that explicitly informs one about the underlying mathematical structures. This also makes it simple to construct dynamical systems.

In presenting the subject we tried to favour generality whenever it was possible and not obscuring us too much from our study of one-dimensional discrete dynamical systems. While we did like the approach of Broer & Takens (2011), we found their selection of topics somewhat esoteric—or, perhaps more correctly, not that compatible with our subject of study. Hence, actually, quite a few definitions in section I.I are based on Brin & Stuck (2004) and de Vries (2014), and revised to match the notation adopted. For simplicity Lyapunov exponents were treated only in the case needed for studying the dynamical systems in hand. The treatment on Lyapunov exponents is based on Medio & Lines (2001) and Chicone (1999).

We also used Hirsch et al. (2004), mainly in the formulation of lemma 1.

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