DISPERSION RELATION FOR PERTURBED FLUID

1. Notation

The n-dimensional Euclidean space is denoted by $\mathbf{E}^n$, for emphasis. The set of real numbers is denoted by $\mathbf{R}$, and the set of complex numbers is denoted by $\mathbf{C}$. The set of square matrices of order n, with values in the field F, is denoted by $\mathcal{M}_n(F)$. The transpose of a matrix A is denoted by A^{T}.

The Einstein summation convention is used throughout, and indices from the beginning (resp. the middle) of the latin alphabet take values from the set $\{1, 2\}$ (resp. the set $\{1, 2, 3\}$).

In differentation we use Euler's notation, with $D_i f(a) := (\partial f/\partial x_i)(a)$.

2. Analysis

The aim of this analysis is to obtain the dispersion relation for perturbed Newtonian fluid in a rotating frame of reference.

Studies in planetary ring dynamics are generally carried out in cylindrical coordinates, as is clearly appropriate, considering the symmetries present in planetary rings. In studies of moonlet wakes, however, curvature terms are usually neglected in order to simplify calculations. For notational convenience, therefore, we adopt Cartesian coordinates throughout our analysis. Due to other approximations made along the way, one may safely identify Cartesian coordinates with cylindrical coordinates via the small-angle approximation.

2.1. **The Continuity Equation.** Let $x = (x_1, x_2, x_3) \in \mathbf{E}^3$ denote the position and let $\rho, u_i : \mathbf{E}^3 \times \mathbf{R} \to \mathbf{R}$ denote the mass density field and the components of the velocity vector, respectively. The functions ρ and u_i depend on the position x and time t, and we assume they are both smooth.

The continuity equation,

$$D_t\rho + D_i(\rho u_i) = 0, \tag{1}$$

implies that

$$0 = \int_{\mathbf{R}} D_t\rho + D_i(\rho u_i)\, dx_3 = D_t\sigma + D_a(\sigma v_a), \tag{2}$$

assuming that $\rho \to 0$ as $x_3 \to \pm\infty$. Here we have defined the surface density,

$$\sigma := \int_{\mathbf{R}} \rho\, dx_3, \tag{3}$$

and the density-weighted average velocity,

$$v_a := \frac{1}{\sigma}\int_{\mathbf{R}} \rho u_a\, dx_3. \tag{4}$$

Note that $\sigma, v_a : \mathbf{E}^2 \times \mathbf{R} \to \mathbf{R}$.

Now consider eq. (2) as a perturbation problem, i.e., assume that $\sigma = \sum_{n=0}^{\infty} \epsilon^n \sigma_n$ and $v_a = \sum_{n=0}^{\infty} \epsilon^n v_{a,n}$, where ϵ is the perturbation parameter. We assume the unperturbed problem (obtained by setting ϵ equal to zero) has a solution described by σ_0 and $v_{a,0}$. Then

$$D_t\sigma_0 + \epsilon D_t\sigma_1 + D_a[\sigma_0 v_{a,0} + \epsilon(\sigma_0 v_{a,1} + \sigma_1 v_{a,0})] = \mathcal{O}(\epsilon^2) \tag{5}$$

and the first-order problem (obtained by setting the coefficient of ϵ equal to zero) is

$$D_t\sigma_1 + D_a(\sigma_0 v_{a,1} + \sigma_1 v_{a,0}) = 0. \tag{6}$$

We let the perturbed fields describe unifrequent plane waves, i.e., make the ansatz

$$(\sigma_1, v_{a,1}) = (\tilde{\sigma}_1, \tilde{v}_{a,1})e^{\mathrm{i}(k_a x_a - \omega t)}, \tag{7}$$

where $\tilde{\sigma}_1, \tilde{v}_{a,1} \in \mathbf{C}$ denote wave amplitudes, $k_a \in \mathbf{C}$ are the wave vector components, and $\omega \in \mathbf{R}$ is the frequency. Allowing the wave vector components k_a to be complex leads to dissipation. Note that physical fields are described by the real parts of the fields. Consequently eq. (6) reduces to

$$\omega\sigma_1 + v_{a,1}(\mathrm{i}D_a\sigma_0 - k_a\sigma_0) + \sigma_1(\mathrm{i}D_a v_{a,0} - k_a v_{a,0}) = 0. \tag{8}$$

We now fix the value of σ_0 so that we may consider it as constant in the vicinity of the corresponding point in $\mathbf{E}^2$; accordingly its spatial derivatives vanish. We also assume the initial fluid velocity is parallel to the x_2-axis, i.e., let $v_{1,0} = 0$. In addition, we assume variations of the initial fluid velocity $v_{2,0}$ in the x_2-direction are negligible so that $D_2 v_{2,0} = 0$. Then eq. (8) reduces to

$$(\omega - k_2 v_{2,0})\sigma_1 - \sigma_0 k_a v_{a,1} = 0. \tag{9}$$

2.2. **The Navier-Stokes Equation.** Let $\phi, p : \mathbf{E}^3 \times \mathbf{R} \to \mathbf{R}$ denote the gravitational potential and the pressure, respectively, and assume these functions are smooth. The Navier-Stokes equation, written in a rotating frame of reference, is

$$D_t u_i + u_j D_j u_i = -D_i\phi - \frac{1}{\rho}D_i p - \frac{1}{\rho}D_j\tau_{ij} - 2\Omega_{ik}u_k - \Omega_{ik}\Omega_{km}x_m, \tag{10}$$

where $\Omega_{ij} := -\epsilon_{ijk}\Omega_k$; here Ω_i denote the angular velocity vector components. The two last terms of eq. (10) are the Coriolis and centrifugal forces, respectively, which appear due to the rotating frame of reference. The components of the viscous stress tensor τ are, for Newtonian fluid,

$$\tau_{ij} = -2\mu D_{(i}u_{j)} + (2\mu/3 - \zeta)\delta_{ij}D_k u_k. \tag{11}$$

Here μ and ζ are the coefficients of shear and bulk viscosity, respectively, and we allow them to depend on the mass density ρ. Moreover, we assume $\zeta \propto \mu$, so that we may write $2\mu/3 - \zeta = \alpha\mu$, with $\alpha \in \mathbf{R}$. Note that τ is symmetric. We let the reference frame rotate about the x_3-axis, in which case $\Omega_{ik}u_k = \|\Omega\|\epsilon_{i3k}u_k$ and $\Omega_{ik}\Omega_{km}x_m = -\Omega^2\delta_{ia}x_a$. We assume the angular velocity $\|\Omega\|$ is constant. With these assumptions eq. (10) reads

$$D_t u_i + u_j D_j u_i = -D_i\phi - \frac{1}{\rho}D_i p - \frac{1}{\rho}D_j\tau_{ij} - 2\|\Omega\|\epsilon_{i3k}u_k + \Omega^2\delta_{ia}x_a \tag{12}$$

with

$$\tau_{ij} = -2\mu D_{(i}u_{j)} + \delta_{ij}\alpha\mu D_k u_k. \tag{13}$$

We wish to proceed in a manner similar to the one used in section 2.1, but, in order to make any progress, we need to make further assumptions.[1] We begin by

Remark 1 (Thin-disk approximation). Let $\rho = \varrho\delta_0$, where ϱ is a smooth function and δ_0 is the Dirac measure at $x_3 = 0$. Then

$$\int_{\mathbf{R}} \rho(x,t)\,dx_3 = \int_{\mathbf{R}} (\varrho\delta_0)(x,t)\,dx_3 = \varrho(x_1, x_2, 0, t)$$

[1] One might be tempted to dodge the integrals like we did in section 2.1, but, to be able to couple the continuity equation and the Navier-Stokes equation, one's definitions must be consistent. However, e.g., $D_t v_a$ and the density-weighted average of $D_t u_a$ are not equal, as a short calculation shows (for any smooth ρ, that is).

and

$$\int_{\mathbf{R}} (\rho f)(x,t)\,dx_3 = \int_{\mathbf{R}} (\varrho\delta_0 f)(x,t)\,dx_3 = (\varrho f)(x_1,x_2,0,t)$$

by definition. It follows from eq. (3) that, for a thin disk, $\sigma(x_1,x_2,t) = \varrho(x_1,x_2,0,t)$ and $F(x_1,x_2,t) = f(x_1,x_2,0,t)$, where F is the density-weighted average of a function f. Thus, e.g., $v_a(x_1,x_2,t) = u_a(x_1,x_2,0,t)$.

We similarly let $p = p'\delta_0$ and $\mu = \mu'\delta_0$, and assume that $u_3(x_1,x_2,0,t) = 0$. We moreover assume both ϕ and u_i are symmetric with respect to the $x_3 = 0$ plane, so that $D_3\phi(x_1,x_2,0,t) = D_3u_i(x_1,x_2,0,t) = 0$. Then eq. (12) reduces to

$$D_t v_a + v_b D_b v_a = -D_a\Phi - \frac{1}{\sigma}D_a P - \frac{1}{\sigma}D_b T_{ab} - 2\|\Omega\|\varepsilon_{a3b}v_b + \Omega^2 x_a \tag{14}$$

with

$$T_{ab} = -2\eta D_{(a}v_{b)} + \delta_{ab}\alpha\eta D_c v_c. \tag{15}$$

Here Φ is the density-weighted average of ϕ, the pressure is $P := \int_{\mathbf{R}} p\,dx_3$ and the coefficient of shear viscosity is $\eta := \int_{\mathbf{R}} \mu\,dx_3$. Before going further we let $\Phi = \Phi_{\mathrm{p}} + \Phi_{\mathrm{d}}$, where the subscripts p and d refer to planet and disk, respectively.

Now consider eq. (14) as a perturbation problem in σ, v_a, Φ_{d}, η and P. Then

$$\begin{aligned}
\mathcal{O}(\epsilon^2) = {}& D_t v_{a,0} + \epsilon D_t v_{a,1} + v_{b,0}D_b v_{a,0} + \epsilon(v_{b,1}D_b v_{a,0} + v_{b,0}D_b v_{a,1}) \\
&+ D_a(\Phi_{\mathrm{p}} + \Phi_{\mathrm{d},0}) + \epsilon D_a\Phi_{\mathrm{d},1} + \frac{1}{\sigma_0}(1 - \epsilon\sigma_1/\sigma_0)D_a P_0 \\
&+ \frac{\epsilon}{\sigma_0}D_a P_1 + \frac{1}{\sigma_0}(\epsilon\sigma_1/\sigma_0 - 1)D_b(2\eta_0 D_{(a}v_{b),0} - \delta_{ab}\alpha\eta_0 D_c v_{c,0}) \\
&- \frac{\epsilon}{\sigma_0}D_b[2\eta_1 D_{(a}v_{b),0} + 2\eta_0 D_{(a}v_{b),1} - \delta_{ab}\alpha(\eta_1 D_c v_{c,0} + \eta_0 D_c v_{c,1})] \\
&+ 2\|\Omega\|\varepsilon_{a3b}(v_{b,0} + \epsilon v_{b,1}) - \Omega^2 x_a
\end{aligned} \tag{16}$$

and the first order problem is

$$\begin{aligned}
0 = {}& D_t v_{a,1} + v_{b,1}D_b v_{a,0} + v_{b,0}D_b v_{a,1} + D_a\Phi_{\mathrm{d},1} + 2\|\Omega\|\varepsilon_{a3b}v_{b,1} \\
&+ \frac{1}{\sigma_0}\big[D_a P_1 - (\sigma_1/\sigma_0)D_a P_0\big] + \frac{\sigma_1}{\sigma_0^2}D_b(2\eta_0 D_{(a}v_{b),0} - \delta_{ab}\alpha\eta_0 D_c v_{c,0}) \\
&- \frac{1}{\sigma_0}D_b[2\eta_1 D_{(a}v_{b),0} + 2\eta_0 D_{(a}v_{b),1} - \delta_{ab}\alpha(\eta_1 D_c v_{c,0} + \eta_0 D_c v_{c,1})].
\end{aligned} \tag{17}$$

We fix the values of σ_0, η_0 and P_0 so that we may consider them as constant in the vicinity of the corresponding point in $\mathbf{E}^2$; accordingly their spatial derivatives vanish. We moreover let $v_{1,0} = 0$ and $D_2 v_{2,0} = 0$, as in section 2.1. With these assumptions eq. (17) simplifies to

$$\begin{aligned}
0 = {}& D_t v_{a,1} + \delta_{a2}v_{1,1}D_1 v_{2,0} + v_{2,0}D_2 v_{a,1} + D_a\Phi_{\mathrm{d},1} + \frac{1}{\sigma_0}D_a P_1 \\
&+ \frac{\sigma_1}{\sigma_0^2}\delta_{a2}\eta_0 D_1^2 v_{2,0} - \frac{1}{\sigma_0}\big[D_1 v_{2,0}(\delta_{a1}D_2 + \delta_{a2}D_1)\eta_1 + \delta_{a2}\eta_1 D_1^2 v_{2,0}\big] \\
&- \frac{\eta_0}{\sigma_0}D_b(2D_{(a}v_{b),1} - \delta_{ab}\alpha D_c v_{c,1}) + 2\|\Omega\|\varepsilon_{a3b}v_{b,1}.
\end{aligned} \tag{18}$$

We let the perturbed fields describe unifrequent plane waves, as in section 2.1 [see eq. (7)]. We moreover let $\Phi_{\mathrm{d},1} = -2\pi G\sigma_1/\|k\|$, $P_1 = P'\sigma_1$ and $\eta_1 = \eta'\sigma_1$, where $P' = (D_\sigma P)_0$ and $\eta' = (D_\sigma\eta)_0$. Here the subscript 0 refers to the unperturbed value. Then eq. (18) reduces to

$$
\begin{aligned}
0 = {} & (\omega - k_2 v_{2,0}) v_{a,1} + \mathrm{i}\delta_{a2} v_{1,1} D_1 v_{2,0} + \gamma k_a \sigma_1 + \mathrm{i}\frac{\sigma_1}{\sigma_0^2}\delta_{a2}\eta_0 D_1^2 v_{2,0} \\
& + \frac{\sigma_1}{\sigma_0}\eta' \big[(\delta_{a1}k_2 + \delta_{a2}k_1) D_1 v_{2,0} - \mathrm{i}\delta_{a2} D_1^2 v_{2,0}\big] + 2\mathrm{i}\|\Omega\|\varepsilon_{a3b} v_{b,1} \\
& + \mathrm{i}\frac{\eta_0}{\sigma_0} k_b (k_a v_{b,1} + k_b v_{a,1} - \delta_{ab}\alpha k_c v_{c,1}),
\end{aligned} \tag{19}
$$

where $\gamma = 2\pi G/\|k\| - P'/\sigma_0$.

2.3. **The Dispersion Relation.** Combining eq. (9) and eq. (19) leads to the homogeneous matrix equation

$$
A(\sigma_1, v_{1,1}, v_{2,1})^{\mathrm{T}} = 0, \tag{20}
$$

where $A \in \mathcal{M}_3(\mathbf{C})$ and

$$
\begin{aligned}
& A_{11} = \omega - k_2 v_{2,0}; \quad A_{12} = -\sigma_0 k_1; \quad A_{13} = -\sigma_0 k_2, \\
& A_{21} = \gamma k_1 + \frac{1}{\sigma_0}\eta' k_2 D_1 v_{2,0}, \\
& A_{22} = \omega - k_2 v_{2,0} + \mathrm{i}\frac{\eta_0}{\sigma_0}\big[(2-\alpha)k_1^2 + k_2^2\big], \\
& A_{23} = \mathrm{i}\frac{\eta_0}{\sigma_0}(1-\alpha)k_1 k_2 - 2\mathrm{i}\|\Omega\|, \\
& A_{31} = \gamma k_2 + \mathrm{i}\frac{\eta_0}{\sigma_0^2} D_1^2 v_{2,0} + \frac{1}{\sigma_0}\eta'(k_1 D_1 v_{2,0} - \mathrm{i}D_1^2 v_{2,0}), \\
& A_{32} = \mathrm{i}D_1 v_{2,0} + \mathrm{i}\frac{\eta_0}{\sigma_0}(1-\alpha)k_1 k_2 + 2\mathrm{i}\|\Omega\|, \\
& A_{33} = \omega - k_2 v_{2,0} + \mathrm{i}\frac{\eta_0}{\sigma_0}\big[(2-\alpha)k_2^2 + k_1^2\big].
\end{aligned} \tag{21}
$$

A nontrivial solution of eq. (20) requires that

$$
\det A = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k} = 0. \tag{22}
$$

This equation is the dispersion relation; it relates the spatial and temporal properties of waves, through k and ω, respectively.